Jackson 5.19(a)

$$\vec\nabla\cdot\vec H = -\vec\nabla\cdot\vec M, \quad \vec\nabla\times\vec H = \vec J_f = 0. \quad \Rightarrow \quad \vec H = \vec\nabla\Phi$$

$$\vec\nabla^2\Phi = \vec\nabla(\vec H) = -\vec\nabla\vec M$$

Φ satisfies poisson eq with "magnetic polarization charge" $\rho = \# \vec\nabla\cdot\vec M$

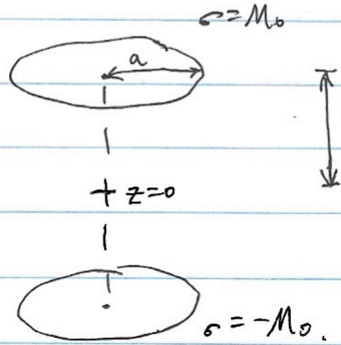
$\vec M = M_0\hat z \quad \Rightarrow \quad \vec\nabla\cdot\vec M = 0$ inside cylinder.

Yet applying Divergence theorem shows that there is $\vec\nabla\cdot\vec M$ at top and bottom surface.

$$\int_V (\vec\nabla\cdot\vec M)\, d\tau = \oint_S \vec M\cdot d\vec a = 0$$

$$= M_0(\pi a^2) - M_0(\pi a^2) = 0.$$

This implies there is "charge density" M_0 at top surface and $-M_0$ at bottom surface.



$$\Phi = \frac{1}{4\pi}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$$

$$= \frac{1}{4\pi}\int \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'|}\, da_{top} + \int \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'|}\, da_{bottom}$$

$$\Phi = \frac{1}{4\pi}\int \frac{r\,d\phi\,dr\; M_0}{\sqrt{\left(\frac{L}{2}-z\right)^2 + r^2}} - \frac{1}{4\pi}\int \frac{r\,d\phi\,dr\; M_0}{\sqrt{\left(\frac{L}{2}+z\right)^2 + r^2}}.$$

$$= \frac{1}{2}\int_0^a \frac{r\,dr\; M_0}{\sqrt{\left(\frac{L}{2}-z\right)^2 + r^2}} - \frac{1}{2}\int_0^a \frac{r\,dr\; M_0}{\sqrt{\left(\frac{L}{2}+z\right)^2 + r^2}}$$

$$= \frac{M_0}{2}\left\{\int_0^a \frac{r\,dr}{\sqrt{\left(\frac{L}{2}-z\right)^2 + r^2}} - \int_0^a \frac{r\,dr}{\sqrt{\left(\frac{L}{2}+z\right)^2 + r^2}}\right\}$$

$$= \frac{M_0}{2}\left\{\sqrt{\left(\frac{L}{2}-z\right)^2 + r^2} - \sqrt{\left(\frac{L}{2}+z\right)^2 + r^2}\right\}_0^a$$

$$\Phi = \frac{M_0}{2}\left[\sqrt{\left(\frac{L}{2}-z\right)^2 + a^2} - \sqrt{\left(\frac{L}{2}+z\right)^2 + a^2} + 2z\right]$$

We obtain $\vec{H}$ from $\vec{H} = \vec{\nabla}\Phi$

Φ has dependence on z, so $H_x = 0$, $H_y = 0$,

$$H_z = \frac{M_0}{2}\left[\frac{\cancel{2}\left(\frac{L}{2} - z\right)}{\sqrt{\left(\frac{L}{2}-z\right)^2 + a^2}}\left(\frac{1}{\cancel{2}}\right)(-1) - \frac{\cancel{2}\left(\frac{L}{2}+z\right)}{\sqrt{\left(\frac{L}{2}+z\right)^2 + a^2}}\left(\frac{1}{\cancel{2}}\right) + 2 \right]$$

$$= \frac{M_0}{\cancel{4}\,2}\left[\frac{\left(\frac{L}{2} - z\right)}{\sqrt{\left(\frac{L}{2}-z\right)^2 + a^2}}(-1) - \frac{\left(\frac{L}{2}+z\right)}{\sqrt{\left(\frac{L}{2}+z\right)^2 + a^2}} + 2 \right]$$

$$= \frac{M_0}{2}\left[2 - \frac{\left(\frac{L}{2} - z\right)}{\sqrt{\left(\frac{L}{2}-z\right)^2 + a^2}} - \frac{\left(\frac{L}{2}+z\right)}{\sqrt{\left(\frac{L}{2}+z\right)^2 + a^2}} \right]$$

From $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$, we obtain $\vec{B} = \mu_0(\vec{H} + \vec{M})$, thus

$$\vec{B} = \begin{cases} \hat{z}\,\frac{\mu_0 M_0}{2}\left[2 - \cdots\cdots - \frac{\left(\frac{L}{2}+z\right)}{\sqrt{\cdots}}\right] & \text{outside cylinder} \\ \\ \hat{z}\,\frac{\mu_0 M_0}{2}\left[4 - \cdots\cdots - \frac{\left(\frac{L}{2}+z\right)}{\sqrt{\cdots}}\right] & \text{inside cylinder.} \end{cases}$$

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